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On maximal resonance of polyomino graphs

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Abstract A polyomino graph is a finite plane 2-connected bipartite graph every interior face of which is bounded by a regular square of side length one. Let k be a positive integer, a polyomino graph G is k-resonant if the deletion of any $i \leq k$ vertex-disjoint squares from G results in a graph either having perfect matchings or being empty. If graph G is k-resonant for any integer $k \geq 1$, then it is called maximally resonant. All maximally resonant polyomino graphs are characterized in this work. As a result, the least integer k such that a k-resonant polyomino graph is maximally resonant is determined.

Keywords Polyomino graph · k-resonance · Maximal resonance

1 Introduction

The concept of resonance originates from the conjugated circuits method which was early found in [30] and [9,10]. Conjugated circuits were also found in Clar's aromatic sextet theory [6] and Randić's conjugated circuit model [21–25]. Klein [13] emphasized the connection of Clar's ideas with the conjugated circuits method. In mathematics [19], a conjugated circuit is named an alternating cycle. A matching (resp. perfect matching) of a graph is a set of its edges such that every vertex of the graph is incident with at most (resp. exactly) one edge in this set. For a graph G with

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a matching M, an M-alternating cycle is a cycle of which the edges appear alternately in and out of M.

This work studies the maximal resonance of polyomino graphs [1], also called square-cell configurations [8] or chess-boards [3], which are finite plane connected bipartite graphs with every interior face being bounded by a regular square of side length one. Let k be a positive integer, a polyomino graph G is k-resonant if delete any $i \leq k$ vertex-disjoint squares from G, i.e., delete the edges and vertices of the squares together with all the edges incident with some of the deleted vertices, the remained graph has a perfect matching or is empty. In other words, G is k-resonant if for any $i \leq k$ vertex-disjoint squares, there is a perfect matching M such that these i squares are all M-alternating cycles. A polyomino graph is called maximally resonant if it is k-resonant for any integer $k \geq 1$. Clearly, polyomino graphs containing cut-vertices are not 1-resonant. And so, all polyomino graphs are assumed 2-connected in what follows.

Polyomino graphs have useful applications in statistical physics and in modeling problems of surface chemistry (please refer to ref. [8] and the references therein). They are also modelings of many interesting combinatorial subjects, such as hypergraphs [1], domination problem [3], rook polynomials [20], etc. In fact, problems based on perfect matchings was extensively studied on fragments of the square-planar net [4,11,15,26,35]. Also, Kivelson developed the conjugated circuits method for the polyomino graphs [12].

On the other hand, k-resonance of molecular graphs have been investigated extensively [7,14,16,17,27,28,31–33,37], but the 2-resonance for benzenoid systems, open-ended nanotubes and carbon fullerenes remains unknown. For advances on maximal resonance of graphs, it is known that if a benzenoid system is 3-resonant then it is maximally resonant [37]. This conclusion is also true for coronoid systems [5], open-end nanotubes [32], toroidal polyhexes [27,34], Klein-bottle polyhexes [28] and fullerene graphs [31], B-N fullerene graphs [33], generalized benzenoid systems and generalized B-N fullerene graphs [18,29].

For a class of graphs, if n is the least integer such that every n-resonant graph in the class is maximally resonant, then to completely characterize the k-resonance of these graphs, it suffices to characterize the k-resonance for every integer $k \leq n$. And so, it is important to determine this least integer n. For 2-connected polyomino graphs, by characterizing all these maximally resonant graphs it is shown in this work that this integer $n = 4$. We shall present the main results of this work in Sect. 2, their proofs are presented in the last section. As preliminaries, the k-resonance of polyomino graphs $P_m \times P_n$ $(m, n \geq 2)$ are presented in Sect. 3.

Before proceeding, we need to introduce some more symbols and terminologies. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A graph G' is a subgraph of G, denoted by $G' \subseteq G$, if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. $|V(G)|$ denotes the number of vertices of G. A catacondensed polyomino graph is a polyomino graph every vertex of which lies on the boundary of the outer face. For other symbols and terminologies not specified herein, we follow that of [2].

2 Main results

Before presenting the main results, we introduce some special classes of graphs. The Cartesian product $P_m \times P_n$ of paths P_m and P_n is called a regular polyomino graph, where $(m, n \geq 2)$. For clarity, regular polyomino graph $P_4 \times P_5$ is shown in Fig. 1. Particularly, if $\{m, n\} \cap \{2\} \neq \emptyset$, we call the regular polyomino graph a ladder graph.

Another special graph G_0 and two possible fragments of graphs are depicted in Fig. 2. With these terminologies we can now list our main results as follows.

Theorem 2.1 *For any given integer $k \geq 4$, a polyomino graph G is k-resonant if and only if either $G \cong G_0$, or each maximal regular polyomino subgraph $P_m \times P_n$ has $\{m, n\} \cap \{2, 4\} \neq \emptyset$ such that the adjacent squares of a $P_4 \times P_n$ $(n \geq 3)$ can only exist on the positions illustrated in B or C of Fig. 2 and any pair of $P_2 \times P_{n_1}$ and $P_2 \times P_{n_2}$ are either disjoint or share exactly one square.*

As a direct consequence of Theorem 2.1, we have the following corollary.

Corollary 2.2 *A polyomino graph is maximally resonant if and only if it is 4-resonant.*

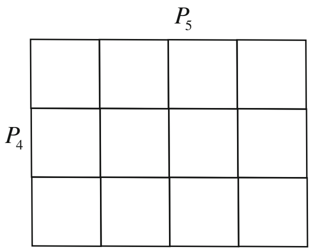


Fig. 1 A regular polyomino graph $P_4 \times P_5$

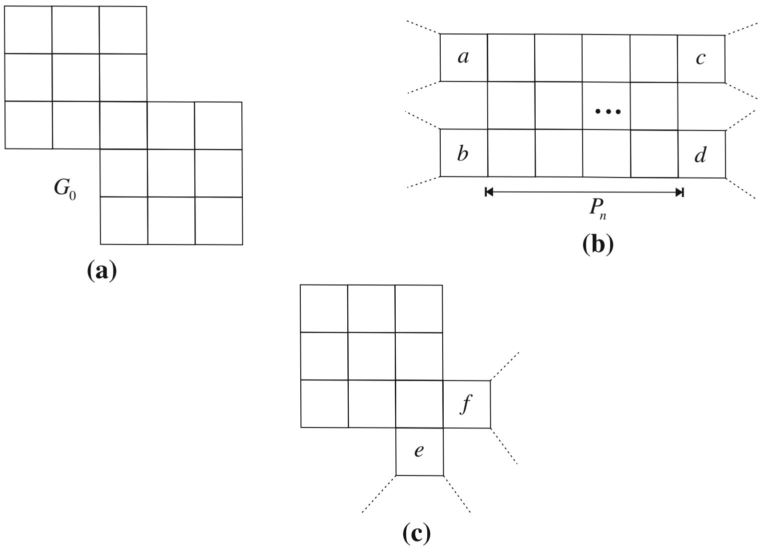


Fig. 2 **a** , G_0. a, b, c, d and e, f are the possible adjacent squares of $P_4 \times P_n$ $(n \geq 3)$ and $P_4 \times P_4$ in (**b**) and (**c**), respectively

Theorem 2.3 *The least positive integer k such that every polyomino graph is maximally resonant if and only if it is k-resonant is* 4.

3 Preliminary

For any positive integer $k \geq 1$, a k-resonant polyomino graph contains perfect matchings. And so, it has an even number of vertices.

Lemma 3.1 *A 2-connected catacondensed polyomino graph G is maximally resonant if and only if any pair of its maximal ladder graphs are either disjoint or share exactly one square.*

Proof Necessity: Suppose that two different maximal ladder graphs L_1 and L_2 of G are neither disjoint nor share one square. Since G is catacondensed and 2-connected, $L_1 \cap L_2$ is an edge, say e. Refer to Fig. 3a where the squares are labeled. Let H_1 be the component of $G - a_1 - b_1$ that contains v and H_2 the union of components of $G - b$ that are subgraphs of H_1. Then $| V(H_2) |=| V(H_1) | -1$. Hence, at least one of H_1 and the components of H_2 is odd. Then G is not maximally resonant.

Sufficiency: We use induction on the number of squares of G. It is not difficult to see that it holds for the trivial case when G is a single square. Suppose G contains more than one square. Let $F \neq \emptyset$ be an arbitrary set of vertex-disjoint squares of G and $f \in F$. Since G does not contain the structure of Fig. 3a, we distinguish the following three cases.

Case 1. f belongs to exactly one maximal ladder graph in G and is disjoint to any other one. Then $G - f$ consists of smaller catacondensed polyomino graphs and independent edges. By the induction hypothesis, $G - F$ has perfect matchings.

Case 2. f belongs to two maximal ladder graphs of G. Similar to case 1, $G - F$ has perfect matchings.

Case 3. f belongs to one maximal ladder graph L but is adjacent to other ones, say possibly L_1 and L_2 as illustrated in Fig. 3b. Then $\{a_1, a_2, a_3, a_4\} \cap F = \emptyset$. Then the

Fig. 3 Illustration for the proof of Lemma 3.1

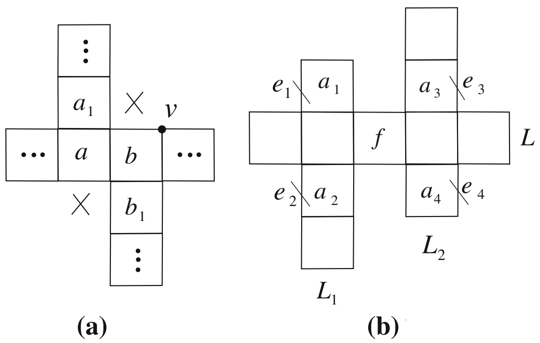
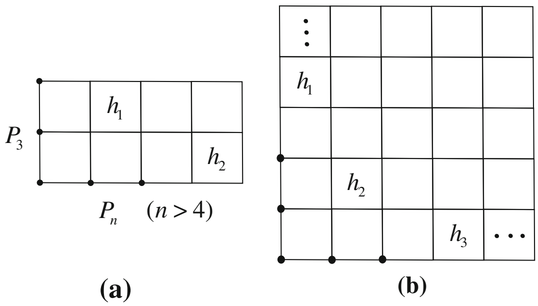


Fig. 4 **a** $P_3 \times P_n$ $(n > 4)$; **b** $P_m \times P_n$ $(m, n > 4)$



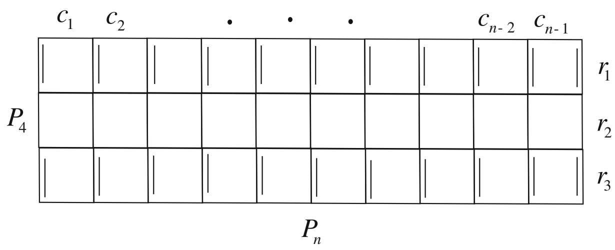


Fig. 5 $P_4 \times P_n$ $(n \geq 3)$ and one of its perfect matchings M

remained graph by cutting edges e_1, e_2, e_3, e_4 in $G - F$ consists of smaller catacondensed polyomino graphs and independent edges. By the induction hypothesis, $G - F$ has perfect matchings.

Since F is arbitrarily chosen, G is maximally resonant. □

In fact, the structure in Fig. 3a is forbidden in all maximally resonant polyomino graphs.

For the maximal resonance of regular polyomino graphs $P_m \times P_n$ $(m, n \geq 2)$, we have the follow conclusion.

Lemma 3.2 *$P_m \times P_n$ $(m, n \geq 2)$ is maximally resonant if and only if $\{m, n\} \cap \{2, 4\} \neq \emptyset$.*

Proof Necessity: Suppose on the contrary that $\{m, n\} \cap \{2, 4\} = \emptyset$. If $m = 3$, since $P_3 \times P_n$ is maximally resonant, it has an even number of vertices. Hence n (> 4) is even. Let h_1 and h_2 be the two squares illustrated in Fig. 4a. Then $P_3 \times P_n - h_1 - h_2$ consists of two odd components, which contradicts the maximal resonance of $P_m \times P_n$. Symmetrically, $n \neq 3$.

If $m, n \geq 5$, let h_1, h_2 and h_3 be the three squares illustrated in Fig. 4b, then $P_m \times P_n - h_1 - h_2 - h_3$ contains a component of five vertices. The necessity follows from this contradiction.

Sufficiency: By Lemma 3.1, it suffices to consider the case when $m = 4$ and $n \geq 3$. Let us label the squares of $P_4 \times P_n$ with three distinct rows and $n - 1$ columns as in Fig. 5, where a perfect matching M is also illustrated by the double edges.

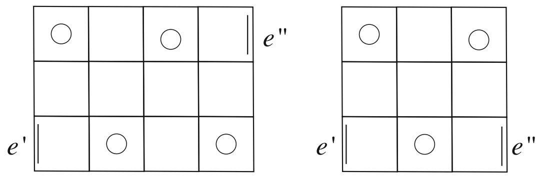


Fig. 6 $H_1 - F$ has a perfect matching $\{e', e''\}$, where the squares inserted cycles belong to F

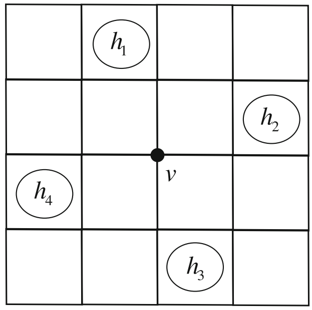


Fig. 7 v is an isolated vertex when delete the four squares inserted cycles

Let F be any set of vertex-disjoint squares of $P_4 \times P_n$, H be the subgraph of $P_4 \times P_n$ induced by all the columns containing at least one square of F and write H'=$P_4 \times P_n - H$. Clearly, every component of H or H' is isomorphic to $P_4 \times P_{n_i}$ for some $n_i \geq 1$. Firstly, $M_0 = M \cap E(H')$ is a perfect matching of H' whenever H' is not empty. Secondly, we shall show in what follows that for any component H_1 of H, either $H_1 - F$ is empty or it also has perfect matchings and so the lemma follows.

If H_1 consists of one column, then $H_1 - F$ is empty or has a perfect matching. Now consider the case when H_1 consists of at least two columns. It is not difficult to see that each column of H_1 contains a unique square of F and that all these squares must lie in row r_1 and row r_3 alternatively as in Fig. 6. No matter whether H_1 has an odd or even number of columns, $H_1 - F$ consists of two disjoint edges e' and e'' as is illustrated in this figure. These two edges enter into a perfect matching of $H_1 - F$. □

Before proving the main results, we remark at first that every polyomino graph is the union of some regular polyomino graphs sharing no common squares.

4 Proof of the main results

Proof of Theorem 2.1: For the necessity, let G be a k-resonant polyomino graph with $k \geq 4$. By Lemma 3.2, we need only consider the case when G is not regular. Let $H_1 \cong P_m \times P_n$ be a maximal regular polyomino subgraph of G with $m, n \geq 3$.

Claim 1 $m < 5$ or $n < 5$.

Suppose to the contrary that $m \geq 5$ and $n \geq 5$. Let h_1, h_2, h_3 and h_4 be the four vertex-disjoint squares in H_1 as shown in Fig. 7. Then $G - h_1 - h_2 - h_3 - h_4$ has an isolated vertex v, which contradicts the condition that G is k-resonant ($k \geq 4$). And so the claim follows.

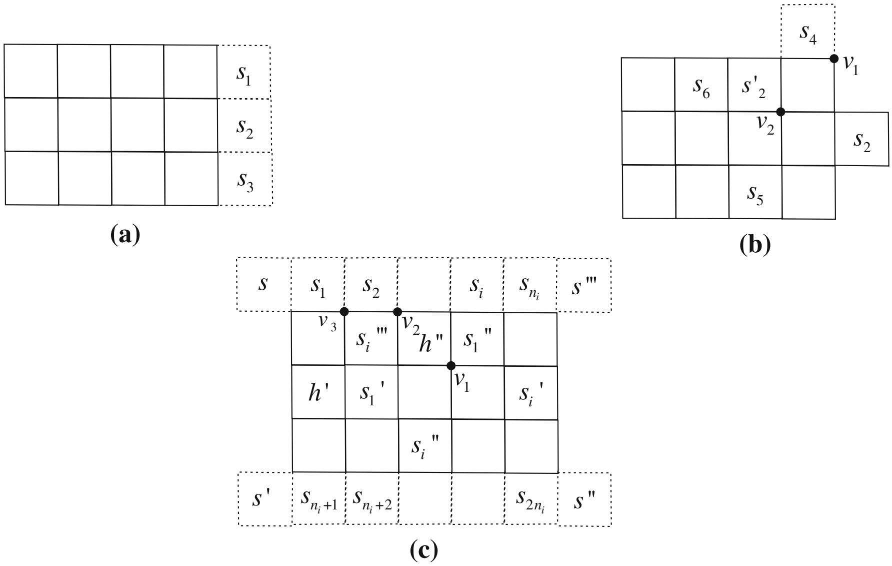


Fig. 8 The illustration for the proof of Claim 2 of Theorem 2.1

Claim 2 If $H_1 \cong P_4 \times P_{n_i}$ for some $n_i \geq 5$, then the squares of G that are adjacent to H_1 can only lie on the four possible positions a, b, c and d as is illustrated in B of Fig. 2.

Since H_1 is a maximal regular polyomino subgraph of G, at least one of the squares $s_1,\ s_2,\ s_3$ illustrated in a of Fig. 8 is not contained in G.

If $s_2 \subseteq G$, assume as we may that $s_1 \nsubseteq G$, then $s_4 \subseteq G$ (refer to b of Fig. 8), since otherwise $G - s_2' - s_2$ would contain an isolated vertex v_1. But then $G - s_2 - s_4 - s_5 - s_6$ contains an isolated vertex v_2. Hence $s_2 \nsubseteq G$.

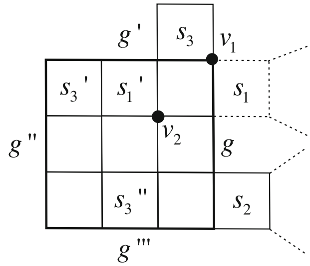
We claim that none of squares $s_2, \ldots, s_{n_i-1}, s_{n_i+2}, \ldots, s_{2n_i-1}$ illustrated in c of Fig. 8 belongs to G. If otherwise $s_i \subseteq G$, then the deletion of squares s_i, s_i', s_i'', s_i''' from G would result in an isolated vertex v_1. If square $s_1 \subset G$, then the deletion of squares $s_1,\ s_1'$, and s_1'' from G results in an isolated vertex v_2. This contradiction shows that $s_1 \nsubseteq G$. Similarly, $s_2, \ldots, s_{2n_i} \nsubseteq G$. If $s \subseteq G$, then the deletion of squares s, h' and h'' (the square containing "v_2") results in an isolated vertex v_3. This contradiction shows that $s \nsubseteq G$. Similarly, $s', s'', s''' \nsubseteq G$. And so, Claim 2 follows.

Claim 3 If $H_1 \cong P_4 \times P_4$, then either $G \cong G_0$ or the possible squares adjacent to H_1 are those illustrated in B or C of Fig. 2.

We shall show at first that none of the squares g, g', g'' and g''' in Fig. 9 is contained in G. If $g \subseteq G$, by the maximality of H_1, at least one of s_1 and s_2, say $s_1 \nsubseteq G$. And so, $s_3 \subseteq G$ since otherwise $G - g - s_1'$ contains isolated vertex v_1 as is illustrated in Fig. 9. But now $G - g - s_3 - s_3' - s_3''$ contains isolated vertex v_2. These contradictions show that $g \nsubseteq G$. Similarly, $g', g'', g''' \nsubseteq G$.

Next we shall consider that which of squares $s_1, s_2, \ldots, s_{11}$ and s_{12} illustrated in a of Fig. 10 are contained in G. Since G is not regular and 2-connected, at least one square of $s_1, s_2, \ldots, s_8$ belongs to G, say s_1.

Fig. 9 The illustration for the proof of Claim 3 of Theorem 2.1



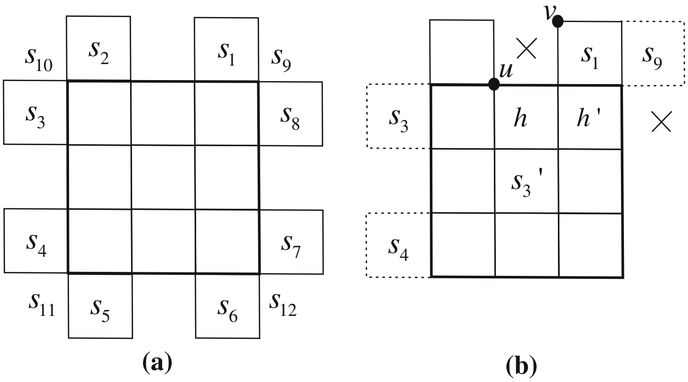


Fig. 10 The illustration for the proof of Claim 3 of Theorem 2.1

Since every component of $G - s_0$ is even, if $s_3 \subseteq G$ then the component of $G - s_1 - s_3 - s_3'$ that contains vertex u has an odd number of vertices (refer to b of Fig. 10). This contradiction shows that $s_3 \nsubseteq G$. If $s_4 \subseteq G$, then the component of $G - s_1 - s_4 - s_3'$ that contains vertex u is an odd component. This contradiction shows that $s_4 \nsubseteq G$. Similarly, one can show that $s_7 \nsubseteq G$.

If $s_8 \nsubseteq G$, then $s_9 \nsubseteq G$ since otherwise the deletion of squares s_9 and h would result in an odd component containing v (refer to b of Fig. 10). Similarly, if $s_2 \subseteq G$ then $s_{10} \nsubseteq G$; if $s_5 \subseteq G$ then $s_{11} \nsubseteq G$; if $s_6 \subseteq G$ then $s_{12} \nsubseteq G$. It follows from these discussions that if $s_8 \nsubseteq G$, then G has the structure B of Fig. 2.

If $s_8 \subseteq G$, then with similar method employed in the above two paragraphs one can show that $s_2, s_5, s_6 \nsubseteq G$. Since G is 2-connected, it follows that $s_{10}, s_{11}, s_{12} \nsubseteq G$. Now, if $s_9 \nsubseteq G$, then G has the structure C of Fig. 2. If $s_9 \subseteq G$, then $s_{13} \subseteq G$ since otherwise $G - s_9 - s_9'$ has the isolated vertex v_1 as is illustrated in a of Fig. 11. Similarly, $s_{14} \subseteq G$. And so, s_{15} or $s_{16} \subseteq G$ since otherwise v_2 is an isolated vertex of $G - s_{13} - s_{14}$ as is illustrated in b of Fig. 11. Assume without loss of generality that $s_{15} \subseteq G$. Then $s_{16} \subseteq G$, since otherwise the component of $G - s_9 - s_9''$ that contains vertices v_3, v_4 and v_5 has an odd number of vertices as illustrated in b of Fig. 11.

If $s_{17}, s_{18} \nsubseteq G$, then $G - s_{14} - s_{15}$ contains an isolated vertex v_6 as illustrated in c of Fig. 11; if $s_{17} \subseteq G$ then $s_1, s_8, s_9, s_{13}, \ldots s_{17}, s_{17}'$ form a subgraph $P_4 \times P_4$ of G, with a similar method to that employed in the first paragraph of the proof of Claim 3 where we show $g \nsubseteq G$, we deduce that $s_{18} \nsubseteq G$; if $s_{18} \subseteq G$ and $s_{17} \nsubseteq G$, then

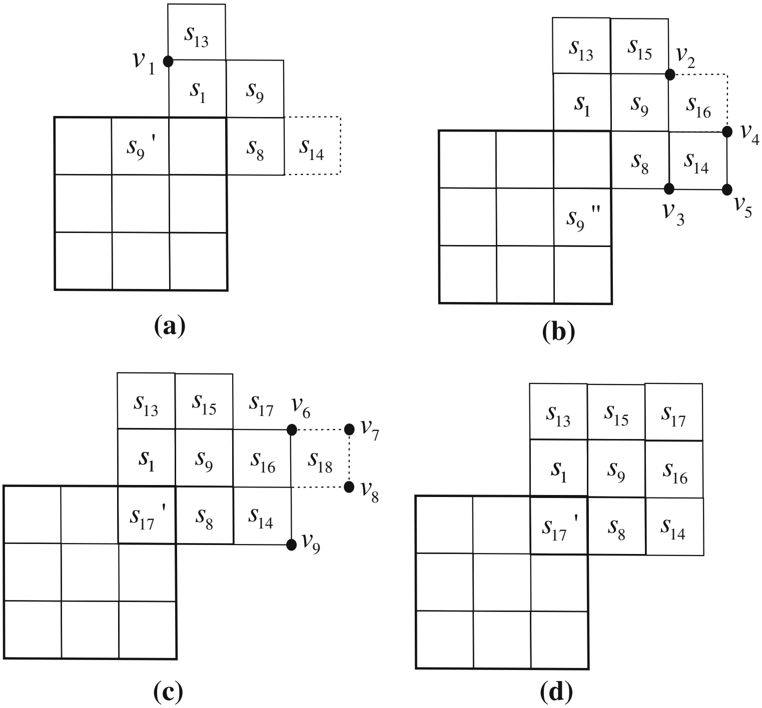


Fig. 11 The illustration for the proof of Claim 3 of Theorem 2.1

$G - s_8 - s_{15}$ or $G - s'_{17} - s_{16}$ contains odd components. These observations show that $s_{18} \nsubseteq G$ and $s_{17} \subseteq G$.

Now, we see that $s_1, s_8, s_9, s_{13}, \ldots s_{17}$ and s'_{17} form a subgraph $H_2 \cong P_4 \times P_4$ of G (refer to d of Fig. 11). Replacing H_1 by H_2 and discussing once more, we deduce that $G \cong G_0$. Hence, Claim 3 follows.

Claim 4 If $H_1 \cong P_3 \times P_{m_1}$ with $m_1 \geq 4$, then $m_1 = 4$ and H_1 has the structure illustrated in B of Fig. 2.

Suppose to the contrary that $m_1 \geq 5$. We shall show at first that none of the squares s_1, s'_1, s''_1 and s'''_1 illustrated in a of Fig. 12 is contained in G. If $s_1 \subseteq G$, by the maximality of H_1 we deduce that $s'_1 \nsubseteq G$. And so, $s_3 \subseteq G$ since otherwise v_1 would be an isolated vertex of $G - s_1 - s_2$ (refer to b of Fig. 12). If the square s_4 illustrated in c of Fig. 12 is not contained in G, then the component of $G - s_1 - s_2$ that contains v_1, v_2, v_3 has an odd number of vertices. This contradiction shows that $s_4 \subseteq G$. And so, either $s_5 \subseteq G$ or $s_6 \subseteq G$ since otherwise the vertex v_4 illustrated in d of Fig. 12 would be either a cut vertex of G or an isolated vertex of $G - s_3 - s''_3$. Combining this observation with Claim 3, we deduce that $s_6 \subseteq G$ and $s_5 \nsubseteq G$. Now, the square s_7 illustrated in e of Fig. 12 is contained in G since otherwise the component of $G - s_3 - s''_3$ that contains vertices v_4, v_5 and v_6 would have an odd number of vertices. Since $s'_1 \nsubseteq G$ and vertex v_1 illustrated in f of Fig. 12 cannot be an isolated vertex of $G - s_1 - s_2 - s_7$,

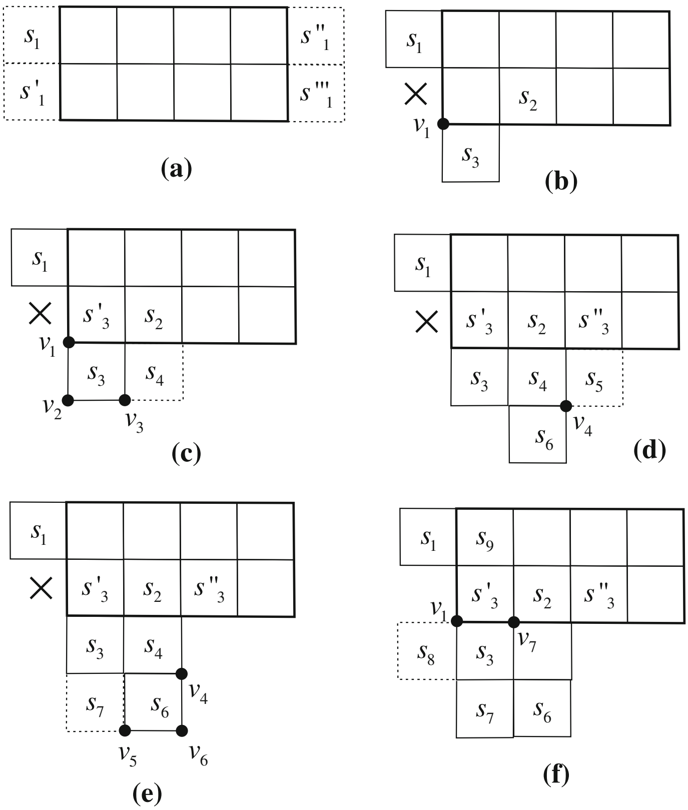


Fig. 12 Illustration for the case of $H_1 \cong P_3 \times P_{m_1}$ $(m_1 \geq 4)$

it follows that $s_8 \subseteq G$. Then $G - s_8 - s_9 - s''_3 - s_6$ has an isolated vertex v_7. These contradictions show that $s_1 \nsubseteq G$. Similarly, $s'_1, s''_1, s'''_1 \nsubseteq G$.

Let us label the squares of H_1 as in a of Fig. 13. $h_1, h_2, \ldots, h_{m_1-1}$ and $h'_1, h'_2, \ldots, h'_{m_1-1}$ are its possible adjacent squares. By Claim 3, we deduce that no three successive squares of $h_1, \ldots, h_{m_1-1}$ or $h'_1, \ldots, h'_{m_1-1}$ belong to G. Suppose that two successive squares of them, say h_i and h_{i+1}, belong to G. Since $m_1 \geq 5$, either $s_{i-1} \subseteq H_1$ or $s_{i+2} \subseteq H_1$, say $s_{i-1} \subseteq H_1$. Then $s_{i-2} \subseteq G$ since otherwise v_1 is an isolated vertex of $G - s'_{i-1} - h_i$. Then either the component of $G - s_{i-1} - h_{i+1}$ containing v_2 or the one of $G - s_{i-2} - s'_i - h_{i+1}$ is odd, as in b of Fig. 13. This contradiction shows that no two successive squares of $h_1, h_2, \ldots h_{m_1-1}$ or $h'_1, h'_2, \ldots h'_{m_1-1}$ belong to G. But now, as is illustrated in c of Fig. 13 the component of $G - S_2 - S'_4$ that contains $u_1, u_2, \ldots u_5$ has an odd number of vertices. This contradiction shows that $m_1 \leq 4$. Therefore $m_1 = 4$.

A similar argument shows that none of the squares s_1, s'_1, s''_1 and s'''_1 illustrated in a of Fig. 14 belong to G when $H_1 = P_3 \times P_4$. Other possible adjacent squares of H_1 are labelled in b of Fig. 14. Since H_1 is a maximal regular grid graph in G, at least one of s_1, s_2 and s_3 is not contained in G. If $s_2 \subseteq G$, assume without loss of generality that

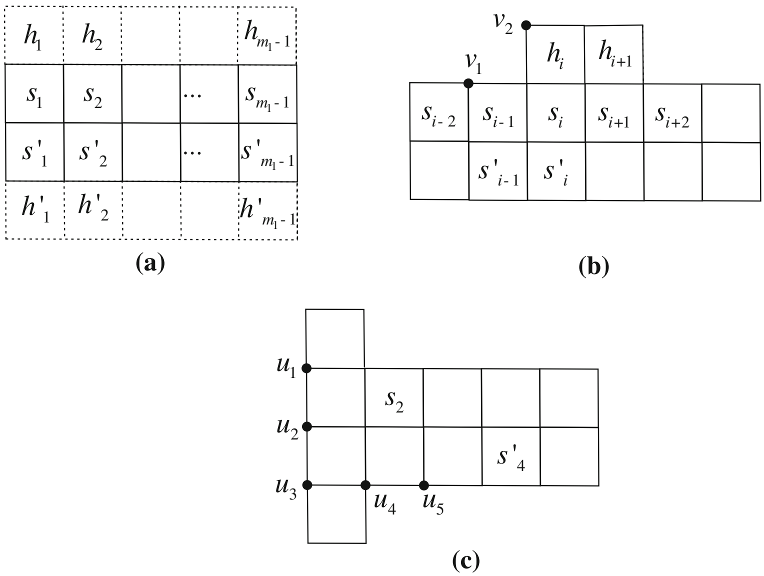
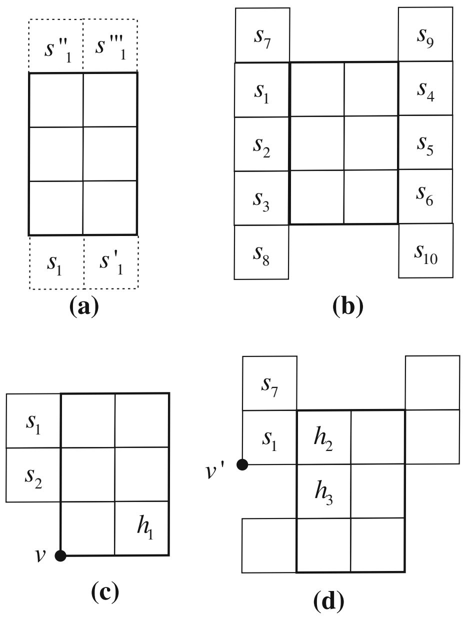


Fig. 13 Illustration for the case of $H_1 \cong P_3 \times P_{m_1}$ $(m_1 \geq 4)$

Fig. 14 Illustration for the case of $H_1 \cong P_3 \times P_4$



$s_3 \nsubseteq G$, then the vertex v illustrated in c of Fig. 14 is an isolated vertex of $G - s_2 - h_1$. This contradiction shows that $s_2 \nsubseteq G$. Similarly, $s_5 \nsubseteq G$.

If $s_7 \subseteq G$, then $s_1 \subseteq G$ since otherwise G would contain a cut vertex. But now, as is illustrated in d of Fig. 14, the component of $G - s_7 - h_3$ that contains the vertex

v' has an odd number of vertices. This contradiction shows that $s_7 \nsubseteq G$. Similarly, $s_8, s_9, s_{10} \nsubseteq G$. Claim 4 follows.

The above four claims and Lemma 3.1 show that every 4-resonant polyomino graph G is either the graph G_0 or the union of some $P_4 \times P_{m_1},\ P_4 \times P_{m_2}, \ldots, P_4 \times P_{m_i}$ $(m_1, m_2, \ldots, m_i \geq 3)$ together with some 4-resonant catacondensed polyomino graphs which connecting them, and the connecting positions satisfy the structures of (B) and (C) for any $P_4 \times P_{m_j}$. And so, the necessity follows.

We continue to prove sufficiency. For any maximal regular polyomino subgraph $P_4 \times P_{m_i}$ with $m_i \geq 3$, if its adjacent squares satisfy the structure (B) then it is called a B-type subgraph of G; if its adjacent squares satisfy the structure (C) then it is called a C-type subgraph of G.

Case 1. $G \ncong G_0$.

Let $P_4 \times P_{m_1}, \ldots, P_4 \times P_{m_n}$ be all the maximal regular polyomino subgraphs of G with $m_1, \ldots, m_n \geq 3$. If $n = 0$, by Lemma 3.1, G is k-resonant for any $k \geq 4$.

If $n = 1$, consider at first the case when $P_4 \times P_{m_1}$ is a B-type subgraph, referring to B of Fig. 2. a, b, c, d are the four possible adjacent squares of $P_4 \times P_{m_1}$. To show that G is k-resonant $(k \geq 4)$ in this case, we use induction on m_1. Let F be an arbitrary set of disjoint squares of G. If $m_1 = 2$, G is catacondensed and by Lemma 3.1 it is k-resonant $(k \geq 4)$. Suppose that G is k-resonant $(k \geq 4)$ for smaller integers. Consider large integer m_1. Delete F firstly. If some of the four a, b, c, d belong to G but not to F, then delete the two edges of the square that connect $P_4 \times P_{m_1}$ and the outside. We conclude by Lemmas 3.1, 3.2 and the induction hypothesis that the resultant graph has perfect matchings. And so, G is k-resonant in this case.

Consider secondly that $P_4 \times P_{m_1}$ is a C-type subgraph of G, which implies that $m_1 = 4$. When we delete k vertex-disjoint squares from G, at least one square of e and f is not deleted, say e. Since the deletion of the two edges of e that connect $P_4 \times P_4$ and another polyomino subgraph results in a catacondensed polyomino subgraph and another subgraph with a B-type $P_4 \times P_4$. By Lemma 3.1 and the observation obtained in the previous paragraph, we deduce that G is k-resonant.

If $n \geq 2$, as is observed in the case when $n = 1$, the deletion of any k vertex-disjoint squares and the edges in squares $a,\ b,\ c,\ d$ that connect $P_4 \times P_{m_1}$ and other polyomino subgraphs results in a graph G' with every component being either an isolated edge, or a k-resonant $(k \geq 4)$ catacondensed polyomino graph, or a polyomino graph with less maximal regular polyomino subgraphs of the form $P_4 \times P_{m_i}$ with $m_i \geq 3$. By induction on n, G' has a perfect matching. And so, G is k-resonant in case 1.

Case 2. $G \cong G_0$.

As is illustrated in Fig. 15, let G_1 and G_2 be the two subgraphs isomorphic to $P_4 \times P_4$ and F be the set of any k vertex-disjoint squares of G.

Subcase 1. Squares $a, b, c, e, f, g \nsubseteq F$. By Lemma 3.2, $G_i - d - F$ has perfect matchings, where $i = 1, 2$. And so, $G - F$ has perfect matchings.

Subcase 2. At least one of the squares a, c, e and g belongs to F. By symmetry, we may assume without loss of generality that $a \subseteq F$. Then $c, d, g \nsubseteq F$. As is shown in

Fig. 15 Illustration for case 2

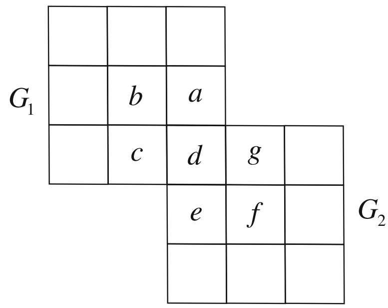
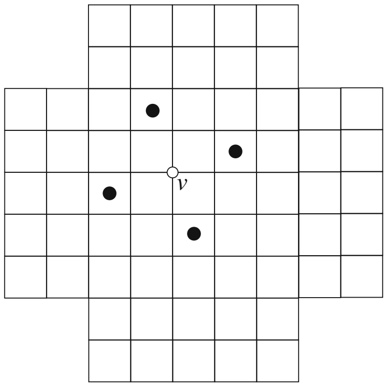


Fig. 16 A 3-resonant but not 4-resonant polyomino graph G



case 1, both $G_1 \cup e - F - e$ and $G_2 \cup a - F - a$ have perfect matchings. And so, $G - F$ has perfect matchings in this case.

Subcase 3. Squares $a, c, d, e, g \nsubseteq F$ but $b \subseteq F$ or $f \subseteq F$. By symmetry, assume without loss of generality that $b \subseteq F$. If $f \subseteq F$, then $F = \{b, f\}$ and $G - F$ has perfect matchings since it is an even cycle in this case; if $f \nsubseteq F$, since both $G_1 - F$ and $G_2 - d - F$ have perfect matchings, $G - F$ also has perfect matchings in this case. And so, sufficiency follows. □

Proof of Theorem 2.3: The graph G illustrated in Fig. 16 is not 4-resonant since the deletion of the four squares containing black points separates an isolated vertex $\{v\}$ from G. By Corollary 2.1, it suffices to show that G is 3-resonant.

Let us delete two arbitrary vertex-disjoint squares C_1 and C_2 from a regular polyomino graph $H = P_{2m} \times P_{2n}$. If these two cycles lie in a same subgraph Q with form $P_2 \times P_{2n}$ or $P_{2m} \times P_2$, then both $G - Q$ and $Q - C_1 - C_2$ have perfect matchings; if otherwise, then $C_1 \subset Q_1$ and $C_2 \subset Q_2$, where Q_1 and Q_2 are two vertex disjoint subgraphs both having form $P_2 \times P_{2m}$ or both having the form $P_{2n} \times P_2$. Clearly, $G - Q_1 - Q_2$, $Q_1 - C_1$ and $Q_2 - C_2$ each contains perfect matchings. And so, we have the following Proposition 4.1.

Proposition 4.1 *For any integers $m, n \geq 1$, the polyomino graph $P_{2m} \times P_{2n}$ is 2-resonant.*

Lemma 4.2 *[36] Let G be a plane bipartite graph with more than two vertices. Then G is 1-resonant if and only if G is elementary.*

Lemma 4.3 *[36] Assume that a 2-connected plane bipartite graph G is weakly elementary. Then G is elementary if and only if the exterior face of G is resonant, i.e., G has a perfect matching M such that the edges of the outer boundary are alternatively in and out of M.*

Lemma 4.4 *[36] Let G be a connected plane bipartite graph that has a perfect matching. If the interior vertices of G (not lying on the boundary of the infinite face of G) all have the same degree then G is weakly elementary.*

By the above three lemmas from Theorems 2.4, 2.10 and 2.11 of [36], we have:

Proposition 4.5 *If the exterior face of a 2-connected polyomino graph G with at least one perfect matching is resonant, then G is 1-resonant.*

As is illustrated in (a) of Fig. 17, let $L_1 = P_{10} \times P_6$, $L_2 = P_6 \times P_{10}$ be two subgraphs of G and $L = L_1 \cap L_2 \cong P_6 \times P_6$. For any three vertex-disjoint squares of G, we shall illustrate that $G - s_1 - s_2 - s_3$ has a perfect matching. The illustrations are depicted in Fig. 17, they have the same labeling as the corresponding distinct cases. These illustrations divide $G - s_1 - s_2 - s_3$ into several vertex-disjoint subgraphs bounded by the bold lines. By Propositions 4.1, 4.2 and Lemmas 3.1, 3.2, these bounded subgraphs have perfect matchings respectively. And so, G is 3-resonant. By the symmetry of L_1 and L_2, the distinct cases can listed as follows.

(a) Squares s_1, s_2, s_3 and graph L are vertex-disjoint.

(b) Exactly one of s_1, s_2 and s_3 intersects L, say s_1.

- (b_1) $s_1 \subseteq L$.
- (b_2) $s_1 \nsubseteq L$.

(c) Exactly two of s_1, s_2 and s_3 intersect L, say s_1 and s_2.

- (c_1) $s_1, s_2 \subseteq L$.
- (c_2) $s_1 \nsubseteq L$ but $s_2 \subseteq L$.
  - (c_{2_1}) $s_1, s_2, s_3 \subseteq L_1$ lie in three adjacent rows.
  - (c_{2_2}) $s_1, s_2, s_3 \subseteq L_1$, and s_1, s_3 lie in two adjacent rows of L_1 but s_2 does not lie in the adjacent row with either of them,
  - (c_{2_3}) $s_1, s_2, s_3 \subseteq L_1$ and only s_1, s_2 lie in two adjacent rows of L_1.
  - (c_{2_4}) $s_1, s_2, s_3 \subseteq L_1$ do not lie in the adjacent rows (may be in the same row).
  - (c_{2_5}) $s_1, s_2 \subseteq L_1$ but $s_3 \subseteq L_2$. The division is the same as that of case c_1.
- (c_3) s_1, s_2 are both adjacent to L, i.e., $s_1, s_2 \nsubseteq L$.
  - (c_{3_1}) $s_1, s_2 \subseteq L_1$ but $s_3 \subseteq L_2$.
  - (c_{3_2}) $s_1, s_2, s_3 \subseteq L_1$.
  - (c_{3_3}) $s_1, s_3 \subseteq L_1,\ s_2 \subseteq L_2$ and s_1, s_2 lie in the adjacent columns.
  - (c_{3_4}) $s_1, s_3 \subseteq L_1,\ s_2 \subseteq L_2$ and s_1, s_2 do not lie in two adjacent columns but s_2, s_3 do.
  - (c_{3_5}) $s_1, s_3 \subseteq L_1,\ s_2 \subseteq L_2$, s_2 does not lie in the adjacent columns with s_1 or s_3 but s_1 and s_3 lie in two adjacent columns.

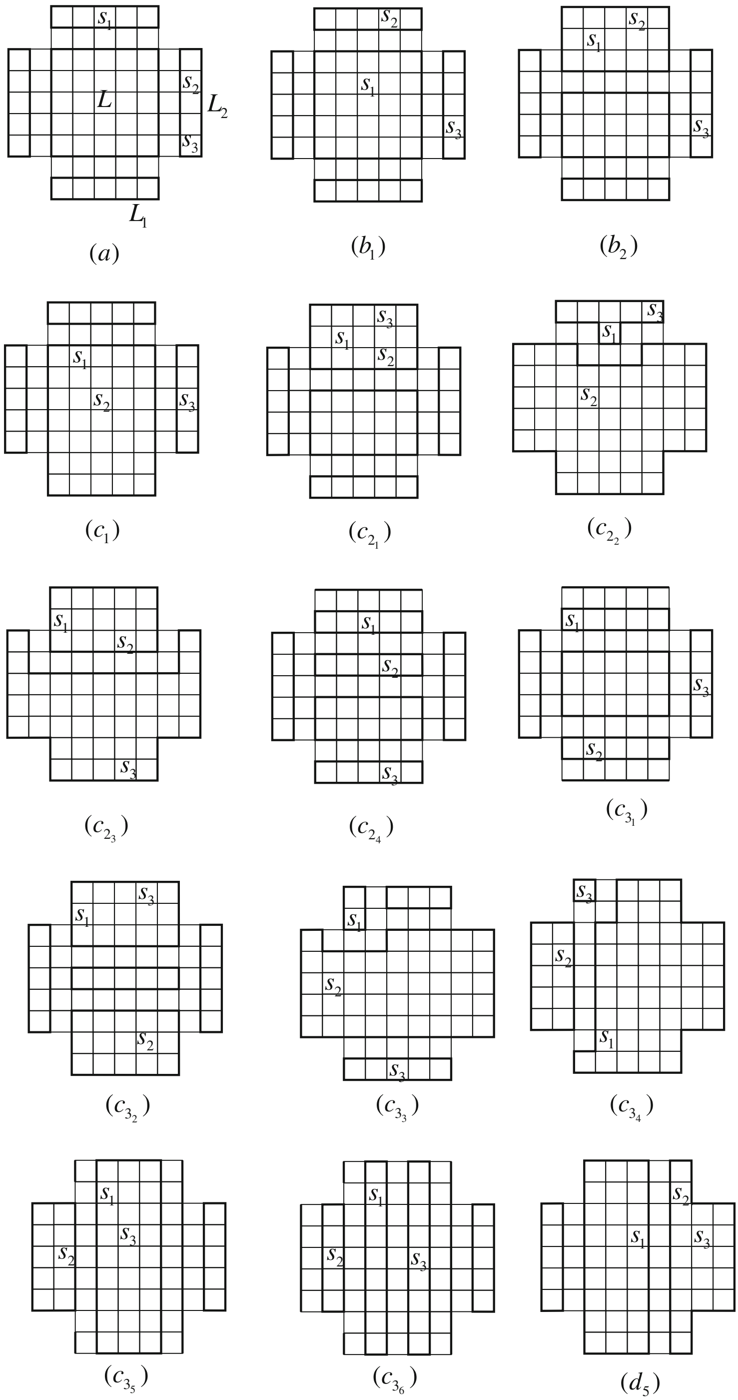


Fig. 17 Illustration for the proof of 3-resonance of G'

(c_{3_6}) $s_1, s_3 \subseteq L_1,\ s_2 \subseteq L_2$, no two of s_1, s_2 and s_3 lie in the adjacent columns.

(d) s_1, s_2, s_3 all intersect L.

(d_1) $s_1, s_2, s_3 \subseteq L_1$ lie in three consequent rows of L_1. The division is similar to case c_{2_1}.

(d_2) $s_1, s_2, s_3 \subseteq L_1$ lie either in the same rows or in disjoint rows. The division is similar to case c_{2_4}.

(d_3) $s_1, s_2, s_3 \subseteq L_1$, exactly two of them lie in adjacent rows. The division is similar to case c_{3_2}.

(d_4) $s_1, s_2 \subseteq L_1,\ s_3 \subseteq L_2$ and $s_2, s_3 \not\subseteq L$. In this case, these three squares can not lie in the consequent three rows or columns. Assume without loss of generality that they are not in three consequent columns. If no two lie in adjacent columns, then the division is similar to case c_{3_6}.

(d_5) $s_1, s_2 \subseteq L_1,\ s_3 \subseteq L_2$ and $s_2, s_3 \not\subseteq L$. If exactly two of them, say s_1 and s_2, lie in adjacent columns. The division is similar to case c_{3_5}. Otherwise, s_1 or s_2 lies in the adjacent column with s_3, then the division is illustrated by d_5 in Fig. 17. □

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